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一、前言

鑒於複變數之利用日益廣泛，其在工程數學的領域裡佔有相當重要的地位，與近十餘年來發展迅速之線性代數相較毫不遜色。且複變數具有其基本特性，它是由一組數值之實數和虛數二部分所組成，具備完整函數之特質；為使一般初學者認識它、瞭解它、應用它，進而發揮它的功能。本文擬從基本性質開始簡介，俾經由對基本性質之瞭解，引發理論分析之思緒，俾應用它解決實際問題。

二、基本性質

一個複變數是由一組數值組成，可分為實數和虛數二部分，若 $Z=(x,y)$，則 x 為 (x,y) 之實數部分，以 $R(Z)=x$ 表示，y 為 (x,y) 之虛數部分，以 $I(Z)=y$ 表示。其具有下列基本性質：

1. $Z=x+iy$，$i=\sqrt{-1}$
2. $Z_1+Z_2=x_1+iy_1+x_2+iy_2=(x_1+x_2)+i(y_1+y_2)$
3. $Z_1-Z_2=(x_1-x_2)+i(y_1-y_2)$
4. $Z_1Z_2=(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$
5. $\dfrac{Z_1}{Z_2}=\dfrac{x_1+iy_1}{x_2+iy_2}=\dfrac{(x_1x_2+y_1y_2)+i(y_1x_2-x_1y_2)}{(x_2)^2+(y_2)^2}$
6. 若 $\bar{Z}$ 為 Z 之共軛複數，則 $Z\bar{Z}=(x+iy)(x-iy)=x^2+y^2$ $\quad\therefore Z\bar{Z}=|Z|^2$
6. (a) $Z+\bar{Z}=(x+iy)+(x-iy)=2x=2R(Z)$，$\quad\therefore x=R(Z)=\dfrac{1}{2}(Z+\bar{Z})$
6. (b) $Z-\bar{Z}=(x+iy)-(x-iy)=2iy=2iI(Z)$ $\quad\therefore y=I(Z)=\dfrac{1}{2i}(Z-\bar{Z})$
7. 若 $Z=\gamma(\cos\theta+i\sin\theta)$，$n$ 為正整數
   則 $Z^n=\gamma^n(\cos n\theta+i\sin n\theta)$，若 $\gamma=1$，則 $Z^n=\cos n\theta+i\sin\theta$
8. $e^{ix}=\cos x+i\sin x$

以上諸性質可應用於解析函數、三角函數、雙曲線函數、指數函數、對數函數、保角寫像，線性分數變換、利曼曲面；並可作積分。複變數亦有數列及級數，如冪級數、泰勒級數，勞倫級數等；而剩餘值亦可積分。複變解析函數可應用於靜電場，二維空間流體運動等。

三、理論分析結果及應用

1. $W=f(Z)$

$$\frac{dW}{dZ}=\lim_{\Delta Z\to 0}\frac{\Delta W}{\Delta Z}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{[u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)]}{\Delta x+i\,y}-\frac{[u(x,y)+iv(x,y)]}{\Delta x+i\Delta y}$$

①若 ΔZ 為實數，則 $\Delta y=0$

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$$\therefore \frac{dW}{dZ} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

②若△Z為虛數，則$\triangle x = 0$，

$$\therefore \frac{dW}{dZ} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

③若 $\frac{dW}{dZ}$ 存在，則

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

∴解析函數具有二個重要條件，即Cauchy - Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

2. $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

①$\therefore \cos Z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

$\therefore \sin Z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

若$x = 0$

則$\cos(iy) = \cosh y$

$\sin(iy) = i \sinh y$

② $\cosh Z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$

$\sinh Z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$

若$x = 0$

則$\cosh(iy) = \cos y$

$\sinh(iy) = i \sin y$

3. $W = f(Z) = u + iV$，　$f(Z) = e^z$

$$f'(Z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{d}{dZ}e^z = e^z, \quad \therefore f'(Z) = u + iV$$

則 $$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iV$$

$$\frac{\partial u}{\partial x} = u, \quad \frac{\partial v}{\partial x} = V$$

若e^z為解析函數，必須滿足 Cauchy - Riemann equation

①$\because f(Z) = u + iv$

$e^z = e^x[(A\cos y + B\sin y) + i(A\sin y - B\cos y)]$

若$y = 0$，$A = 1$，$B = 0$

$\therefore\ e^{z} = e^{x+iy} = e^{x}(\cos y + i \sin y)$

$e^{iy} = \cos y + i \sin y$

② $Z = re^{i\theta}$

$\overline{Z} = re^{-i\theta}$

$Z_1 Z_2 = r_1 r_2 \exp[\, i(\theta_1 + \theta_2)\,]$

$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \exp[\, i(\theta_1 - \theta_2)\,] \quad (r_2 \neq 0)$

③ $Z = re^{i\theta}$

$\ell n Z = \ell n r + i\theta$

$i^{i} = e^{-\frac{\pi}{2}}$ （表示主值）

4. 由Cauchy－Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$$

$$\frac{\alpha^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

u = harmonic ， v = conjugate

5. $\theta = \phi$

$Z(t) = x(t) + iy(t)$

$W = f(Z) = f(Z(t))$

$W = F(t)$

$W = Z^2 = (x^2 - y^2) + i2xy$

$\quad = (x(t))^2 - (y(t))^2 + i\,2x(t)y(t)$

平面上所有曲線皆可列入轉換，惟應保持交角相同

6. $w(t) = f(z(t))]$

$\frac{dw}{dt} = \frac{dt}{dz} \cdot \frac{dz}{dt} \qquad f'(z) \neq 0, \quad w(t) \neq 0$

$\arg \dot{w}(t) = \arg f(z) + \arg \dot{z}(t)$

7. Extended complex plane

$R = \{ x \mid -\infty < x < \infty \}$

$R^{*} = R U \{\infty\} \cup \{-\infty\}$ Extended real number

8. $W = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$

$W' = \frac{ad - bc}{(cz+d)^2} \qquad W' \neq 0$

① special case

$W = z + b$

$W = az$

$W = \frac{1}{z}$

$W = az + b$

② $Z = W = \frac{az+b}{cz+d}$

$cz^2 + dz - az - b = 0$

$cz^2 = (a-d)z - b = 0$ 有二點(fixed points)

若 $c = b = 0$ ， $d = a$ ，則有三點(fixed points)以上

9. $W = Re^{i\phi} = e^{x+iy} = e^x e^{iy}$

$R = e^x$ ， $\phi = y$

Region $e^a \le R \le e^b$ ， $c \le \phi \le d$

10. $W = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y$ ， $v = \cos x \sinh y$

$$\sin x = \frac{u}{\cosh y} \text{ ， } \sin^2 x = \left(\frac{u}{\cosh y}\right)^2$$

$$\cos x = \frac{v}{\sinh y} \text{ ， } \cos^2 x = \left(\frac{v}{\cosh y}\right)^2$$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = \sin^2 x + \cos^2 x = 1$$

11. Riemann surfaces

① $w = \sqrt{2}$ ， $z \neq 0$

② $w = \sqrt[n]{z}$ ， $n = 3, 4, 5, \cdots\cdots$

③ $w = \ell n z = \mathrm{Ln}\, z + 2n\pi i$ ， $n = 0, \pm 1, \pm 2, \cdots\cdots, z \neq 0$

④ $w = z + \frac{1}{z}$ $z \neq 0$

$$w' = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2}$$

⑤ $w = u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

$$u = \left(r + \frac{1}{r}\right)\cos\theta \text{ ， } v = \left(r - \frac{1}{r}\right)\sin\theta$$

12. $F(t) = n(t) + iv(t)$

$\int_a^b F(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$

Real Part 爲 $\int_a^b F(t)dt = \int_a^b \mathrm{Re}[F(t)]dt$

Imaginary Part 爲 $i\int_a^b F(t)dt = i\int_a^b \mathrm{Im}[F(t)]dt$

$\int_a^b rF(t)dt = r\int_a^b F(t)dt$

①若 $re^{i\theta} = \int_a^b F(t)dt$

則 $re^{i\theta} \cdot e^{-i\theta} = e^{-i\theta}\int_a^b F(t)dt = \int_a^b e^{-i\theta}F(t)dt$

$\left|\int_a^b F(t)dt\right| \le \int_a^b |F(t)dt|$ ， $b \ge a$

② $x = x(t)$ ， $y = y(t)$ ， $a \le t \le b$

$x(t)$ 與 $y(t)$ 爲含 t 的連續函數

$z(t)=x(t)+iy(t)$，故 $z(t)$ 也是連續函數，若 c 不會自己打結成相交之路徑，則爲簡單曲線。

③若 c 爲簡單曲線，但頭尾二點相同，$z(b)=z(a)$，則爲簡單封閉曲線（simple closed curve）。

④若 $x(t)$，$y(t)$ 爲可微分函數，則 $z(t)$ 亦爲可微分函數

$z'(t)=x'(t)+iy'(t)$，$a\leq t\leq b$

⑤若 $z'(t)\neq 0$，$a\leq t\leq b$，則 c 爲平滑曲線

$|z'(t)|=\sqrt{[x'(t)]^2+[y'(t)]^2}$

⑥$\int_c z\,dz=2\pi i$

13. Cauchy 積分定理。

①$\int_c z\,dz=0$，$\int_c z^2 dz=0$

$\therefore \oint f(z)\,dz=0$

②$\oint_{c_2} f(z)\,dz=\oint_{c_2} f(z)\,dz$

則 C_1 與 C_2 包圍相同之奇異點

$\therefore \oint_{c_1} f(z)dz-\oint_{c_2} f(z)\,dz=0$

③ $f(z)$ 在簡單封閉曲線C內，且C爲解析函數，Z_0 爲C內任一點，則

$$f(z_0)=\frac{1}{2\pi i}\oint_c \frac{f(z)dz}{z-z_0}$$

14. $$f(z)=\frac{1}{2\pi i}\oint_c \frac{f(s)ds}{s-z}$$

① $$f'(z)=\frac{1}{2\pi i}\int_c \frac{f(s)ds}{(s-z)^2}$$

② $$f''(z)=\frac{1}{\pi i}\int_c \frac{f(s)ds}{(s-z)^3}$$

$\vdots$

③ $$f^{(n)}(z)=\frac{n!}{2\pi i}\int_c \frac{f(s)ds}{(s-z)^{n+1}} \qquad (n=3,4,\cdots\cdots)$$

$\therefore f(z)$ 在某點爲解析函數，則其所有的微分項在該點均爲解析函數。

15. $\lim\limits_{n\to\infty} Z_n=Z$ 爲收斂

級數可用下列各式表示

① $\sum\limits_{n=1}^{\infty} W_n=W_1+W_3+\cdots\cdots$

②$S_n=W_1+W_2+W_3+\cdots\cdots+W_n$

③$R_n=W_{n+1}+W_{n+2}+W_{n+3}+\cdots\cdots$

④ $\sum\limits_{n=1}^{\infty}|W_n|=|W_1|+|W_2|+\cdots\cdots$

若 $W_1+W_2+\cdots\cdots$ 爲收斂級數，則 $\lim\limits_{n\to\infty} W_n=0$

16. ①$|Z_n-a|<\in$ $\quad \in>0$

則 a 稱爲數列 Z_1，Z_2，……之極限點

②$|Z_n-Z_m|<\in$，$n>N$，$m>N$

則數列 Z_1，Z_2，……爲收斂

③$|W_{n+1}+W_{n+2}+\cdots\cdots+W_{n+p}|<\in$，$n>N$，$P=1$，$2\cdots\cdots$

則 $|W_{n+1}+W_{n+2}+\cdots\cdots+W_{n+p}| \leq |W_{n+1}|+|W_{n+2}|+\cdots\cdots+|W_{n+p}|$

17. $u_1 \geq u_2 \geq u_3 \geq \cdots\cdots\cdots$， $\lim\limits_{n\to\infty} u_n = 0$

則 $|R_n| \leq u_{n+1}$

$u_{2n+1} \geq R_{2n} \geq 0$

18. 收歛和發散級數可用下列方法試驗

①Σa_n converges， $0 \leq b_n < a_n \Rightarrow \Sigma b_n$ converges

②Σa_n converges， $\lim \dfrac{b_n}{a_n} = c$ ， $b_n, a_n \geq 0$ ， $\Rightarrow \Sigma b_n$ converges

③$\Sigma |a_n|$ converges $\Rightarrow \Sigma a_n$ converges

④Σa_n ratio， $\lim \left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$ converges

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = L > 1 \quad \text{diverges}$$

⑤Σa_n root test $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L < 1$ converges

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1 \quad \text{diverges}$$

19. Power series 應討論 converge 才有用

① $\sum\limits_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_m x^m + \cdots\cdots$

$$\sqrt[n]{a_n x^n} = \sqrt[n]{a^n} \; |x| \leq 1$$

$$|x| < \frac{1}{\sqrt[n]{a_n}}$$

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| \; |x| \leq 1$$

② $\sum\limits_{m=0}^{\infty} C_m (z-a)^m = C_0 + C_1 (z-a) + C_2 (z-a)^2 + \cdots\cdots$

③ $\sum\limits_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots\cdots$

④ $\sum\limits_{n=0}^{\infty} n!\, z^n = 1 + z + 2z^2 + 6z^3 + \cdots\cdots$

⑤ $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots\cdots + x^n + \cdots\cdots$

20. Taylar series

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z^*)}{z^*-z}\, dz^*$$

$$\frac{1}{z^*-z} = \frac{1}{z^*-a-(z-a)} = \frac{1}{(z^*-a)\left(1-\dfrac{z-a}{z^*-a}\right)}$$

$$\left|\frac{z-a}{z^*-a}\right| = \frac{z-a}{z^*-a} < 1$$

$$= \frac{1}{z^* - a} \left(1 + r + r^2 + \cdots\cdots + r^n + \frac{r^{n+1}}{1-r} \right)$$

$$= \frac{1}{z^* - a} \left(1 + \frac{z-a}{z^*-a} + \left(\frac{z-a}{z^*-a} \right)^2 + \cdots\cdots \right)$$

$$= \frac{1}{2\pi i} \oint \frac{f(z^*)}{z^*-a} dz + \frac{(z-a)\int f(z^*)\, dz^*}{(z^*-a)^2} + \cdots\cdots$$

$$= f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots\cdots$$

$$= \frac{1}{2\pi i} \int \left(\frac{z-a}{z^*-a} \right)^{n+1} / z^* - z \,)\, f(z^*)\, dz^*$$

21. $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots\cdots$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\cdots$$

$$L_n (1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots\cdots, \quad |z| < 1$$

$$-L_n (1-z) = L_n \frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \quad |z| < 1$$

$$L_n \frac{1+z}{1-z} = 2 \left(2 + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\cdots \right), \quad |z| < 1$$

$$\frac{1}{1+z^2} = \frac{1}{1-(-z)^2} = 1 - z^2 + z^4 - z^6 + \cdots\cdots, \quad |z| < 1$$

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots\cdots, \quad |z| < 1$$

$$\tan z = z + \frac{z^3}{3} + \frac{2}{15} z^5 + \frac{17}{315} z^7 + \cdots\cdots, \quad |z| < \frac{\pi}{2}$$

$$\sum_{n=0}^{\infty} f_n(z) = f_0(z) + f_1(z) + f_2(z) + \cdots\cdots$$

$$\sum_{n=0}^{\infty} \int_C f_n(z)\, dz = \int_C f_0(z)\, dz + \int_C f_1(z)\, dz + \cdots\cdots$$

22. Laurent series

① $$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n + \sum_{n=1}^{\infty} \frac{c_n}{(z-a)^n} = b_0 + b_1 (z-a) + b_2 (z-a) + \cdots\cdots$$

$$+ \frac{c_1}{z-a} + \frac{c_2}{(z-a)^2} + \cdots\cdots$$

$$\because b_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^*-a)^{n+1}} dz^*$$

$$c_n = \frac{1}{2\pi i} \int_C (z^*-a)^{n-1} f(z^*)\, dz^*$$

② $f(z)=\frac{1}{2\pi i}\int\frac{f(z^*)}{z^*-z}\,dz^*=\frac{1}{2\pi i}\oint_{c_1}\frac{f(z^*)\,dz^*}{z^*-z}-\frac{1}{2\pi i}\oint_{c_2}\frac{f(z^*)\,dz^*}{z^*-z}$

$$\frac{1}{z^*-z}=\frac{1}{(z^*-a)-(z-a)}=\frac{1}{(z^*-a)}\cdot\frac{1}{1-\frac{z-a}{z^*-a}}=\frac{1}{(z^*-a)}(1+r+r^2+r^3+\cdots+r^n+\cdots\cdots)$$

$$\frac{1}{2\pi i}\oint\frac{z-a}{(z^*-a)}f(z^*)\,dz^*=\frac{z-a}{2\pi i}\oint\frac{f(z^*)\,dz^*}{(z^*-a)^2}=(z-a)f'(a)$$

$$\frac{1}{(z^*-a)-(z-a)}=\frac{1}{z-a}\cdot\frac{1}{1-\frac{z^*-a}{z-a}}=\frac{-1}{z-a}(1+r_1+r_2{}^2+\cdots\cdots+r_1^n+\cdots\cdots)$$

$$\because r_1=\frac{z^*-a}{z-a}$$

$$=\frac{1}{2\pi i}\oint_{c_1}\frac{1}{z^*-a}\left(1+\frac{z-a}{z^*-a}+\left(\frac{z-a}{z^*-a}\right)^2+\cdots+\left(\frac{z-a}{z^*-a}\right)^n\right)$$

$$f(z^*)\,dz^*-\frac{1}{2\pi i}\oint_{c_2}\frac{1}{z-a}\left(1+\frac{z^*-a}{z-a}+\left(\frac{z^*-a}{z-a}\right)^2+\cdots\cdots\right.$$

$$\left.+\left(\frac{z^*-a}{z-a}\right)^n\right)f(z^*)\,dz^*$$

$$=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2$$

$$-\frac{1}{2\pi i}(z-a)^{-1}\oint_{c_2}f(z^*)\,dz^*-\frac{1}{2\pi i}(z-a)^{-2}\oint_{c_2}(z^*-a)$$

$$f(z^*)\,dz^*$$

③ $e^x=1+\frac{1}{1!}(x-0)+\frac{1}{2!}(x-0)^2+\cdots\cdots+\frac{1}{n!}(x-0)^n+R_n(x)$

$$=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\cdots+\frac{x^n}{n!}+\frac{e^z x^{n+1}}{(n+1)!}$$

④ $f(z)=\frac{1}{1-z^2}=\frac{1}{(1+z)(1-z)}=\frac{-1}{(z-1)(1+z)}$, at $z=1$

$$\frac{1}{z+1}=\frac{1}{2+(z-1)}=\frac{1}{2}\cdot\frac{1}{\left(1-\frac{z-1}{2}\right)}=\frac{1}{2}\cdot\frac{1}{1-q}=\frac{1}{2}(1+q+q^2+\cdots+q^n+\cdots\cdots)$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z-1}{2}\right)^n$$

$$\left|\frac{z-1}{2}\right|<1\ ,\quad 0<|z-1|<2$$

$$f(z)=\frac{1}{z-1}\cdot\frac{1}{z+1}=-\frac{1}{2(z-1)}+\frac{1}{4}-\frac{(z-1)}{8}+\frac{(z-1)^2}{16}-+\cdots\cdots$$

⑤ $f(z)=\frac{1}{1-\left(-\frac{z}{z-1}\right)}\cdot\frac{1}{z-1}=\frac{1}{z-1}\left[1+\left(\frac{-2}{z-1}\right)+\left(\frac{-2}{z-1}\right)^2+\cdots\cdots\right]$

$$\left|\frac{2}{z-1}\right|<1\ ,\quad \therefore |z-1|>2$$

$$f(z)=\frac{-1}{(z-1)^2}\left[1+\left(\frac{-2}{z-1}\right)+\left(\frac{-2}{z-2}\right)^2+\cdots\cdots\right]$$

⑥ Singularities：無定義或不存在，無法隔離

zeros：放數值使函數＝0

Isolate：除本身點外，划一半徑只有本身一點，即孤立。

解析函數 $f(z)\ (\neq 0)$ 之 zeros 爲 isolate pole，$z=a$ 爲 pole.

23. Residues

$\int_c f(z)dz=0$

$$f(z)=\sum_{n=0}^{\infty} b_n(z-a)^n+\frac{C_{-2}}{z-a}+\frac{C_{-2}}{(z-a)^2}+\cdots\cdots,\quad 0<|z-a|<R$$

$$\therefore C_1=\frac{1}{2\pi i}\int_c f(z)dz$$

$\int_c f(z)dz=2\pi i C_1$

$$C_1=\operatorname*{Res}_{z=a} f(z)$$

① $f(z)=z^{-4}\sin z=\dfrac{1}{z^3}-\dfrac{1}{3!z}+\dfrac{z}{5!}-\dfrac{z^3}{7!}+\cdots\cdots$

$$\int_c \frac{\sin z}{z^4}dz=2\pi i\, c_1=-\frac{\pi i}{3}$$

② $f(z)=\dfrac{c_1}{z-a}+b_0+b_1(z-a)+b_2(z-a)^2+\cdots\cdots,\quad 0<|z-a|<R\ ，c\neq 0$

$$(z-a)f(z)=c_1+(z-a)[b_0+b_1(z-a)+\cdots]$$

$$\operatorname*{Res}_{z=a} f(z)=c_1=\lim_{z\to a}(z-a)f(z)$$

③ $f(z)=p(z)/q(z)$

$$q(z)=(z-a)\,q'(a)+\frac{(z-a)^2}{2!}q''(a)+\cdots$$

$$\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}(z-a)\frac{p(z)}{q(z)}=\lim_{z\to a}\frac{(z-a)\,p(z)}{(z-a)[q'(a)+(z-a)q''(a)/2+\cdots]}$$

$$\therefore \operatorname*{Res}_{z=a} f(z)=\operatorname*{Res}_{z=a}\frac{p(z)}{q(z)}=\frac{p(a)}{q'(a)}$$

④ $\displaystyle\int_c f(z)dz=2\pi i\sum_{j=1}^{m}\operatorname*{Res}_{z=a} f(z)$

$$\int_c\frac{dz}{(z^3-1)^2}=2\pi i\operatorname*{Res}_{z=1}\frac{1}{(z^3-1)^2}=2\pi i\left(-\frac{2}{9}\right)=-\frac{4\pi i}{9}$$

⑤ $\displaystyle\int_{-\infty}^{\infty} f(x)dx=\lim_{a\to-\infty}\int_a^0 f(x)dx+\lim_{b\to\infty}\int_0^b f(x)dx$

$$\int_{-r}^{r} f(x)dx=2\pi i\sum \operatorname{Res} f(z)-\int_C f(z)dz$$

$$\because |f(z)|<\frac{k}{|z|^2}\ ,\quad |z|=r>r_0$$

$$\left|\int_S f(z)dz\right|<\frac{k}{r^2}\pi r=\frac{k\pi}{r}\ ,\quad r>r_0$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx=2\pi i\sum\operatorname{Res} f(z)$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{2\pi i}{4}\left(-e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}\right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

⑥ $\int_{-\infty}^{\infty} f(x)\cos sx\,dx$, $\int_{-\infty}^{\infty} f(x)\sin sx\,dx$, s real

$$\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = 2\pi i \sum \mathrm{Res}\,[\, f(z)\, e^{isz}\,] \qquad s>0$$

$$\int_{-\infty}^{\infty} f(x)\cos x\,dx = -2\pi \sum I_m\ \mathrm{Res}\,[\, f(z)\, e^{isz}\,], \qquad s>0$$

$$\int_{-\infty}^{\infty} f(x)\sin sx\,dx = 2\pi \sum \mathrm{Re}\ \mathrm{Res}\,[\, f(z)\, e^{isz}\,], \qquad s>0$$

24. $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ potential u

① $u_1 = -c\,\ell n\,|z - x_1|$ potential source lines

$u_2 = c\,\ell n\,|z - x_2|$ 〃

$F_1(z) = -c\,\ell n\,(z - x_1)$ Real parts of the complex potential

$F_2(z) = c\,\ell n\,(z - x_2)$ 〃

$F(z) = F_1(z) + F_2(z) = c\,\ell n\,\frac{z - x_2}{z - x_1}$ 〃

$u = \mathrm{Re}\,F(z)\ c\,\ell n\,\frac{|z - x_2|}{|z - x_1|} = \mathrm{const}$ 〃

$V = I_m F(z) = c\,\arg\frac{z - x_2}{z - x_1} = c\,[\arg(z - x_2) - \arg(z - x_1)) = \mathrm{const}$

$V = c\,(\theta_2 - \theta_1) = \mathrm{const}$

② $V = V_1 + i V_2$ streamline

$\int_C V_t\, ds$ circulation

$V_t = |V| \cos\alpha$ mean velocity

$\frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds}$, $z(s) = x(s) + iy(s)$

$V_t\, ds = V \cdot dz = V_1\, dx + V_2\, dy$ $(dz = dx + i\,dy)$

$\int_C (V_1 dx + V_2 dy) = \iint_D \left(\frac{\partial V_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dxdy$

③ $W_0 = \frac{1}{\pi r^2}\int_D\int \frac{1}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dxdy$

$W = \frac{1}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$ rotation

$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$ irrotational flows

$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$ incompressible

$\int_C (v_1 dx + v_2 dy)$

$\phi(x, y) = \int_{(a,b)}^{(x,y)} (v_1 dx + v_2 dy)$ velocity potential

$v_1 dx + v_2 dy = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy$

$$v = \frac{\partial \phi}{\partial x}, \quad v_2 = \frac{\partial \phi}{\partial y}$$

$$v = v_1 + i v_2 = \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y}$$

The curves $\phi(x, y) = \text{const}$ are called equipotential lines. $\psi(x, y) = \text{const.}$ are the streamlines of the flow. But $\psi(x, y)$ be a conjugate harmonic function of $\phi(x, y)$

④ $F(z) = \phi(x, y) + i\psi(x, y)$ complex potential of the flow

$$F'(z) = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y} = v_1 - i v_2$$

$$v = v_1 + i v_2 = F'(z)$$

25. $$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

$$z = z_0 + re^{i\phi}$$

$$z - z_0 = re^{i\phi}, \quad dz = ire^{i\phi} d\phi$$

$$\therefore f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\phi}) d\phi$$

① $$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{z^* - z} dz^*$$

$$z^* = Re^{i\phi}, \quad 0 \leq \phi < 2\pi$$

$$f(z) = u(r, \theta) + iv(r, \theta) \qquad (z = re^{i\phi})$$

$$dz^* = iRe^{i\phi} d\phi = iz^*$$

$$\therefore f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z^*) \frac{z^*}{z^* - z} d\phi \qquad (z^* = Re^{i\phi}, z = re^{i\theta})$$

$$z = z^* \bar{z}^* / \bar{z} \qquad |z| \leq R$$

$$0 = \frac{1}{2\pi i} \int_C \frac{f(z^*)'}{z^* - z} dz^* = \frac{1}{2\pi} \int_0^{2\pi} f(z^*) \frac{z^*}{z^* - z} d\phi$$

$$0 = \frac{1}{2\pi} \int_0^{2\pi} f(z^*) \frac{\bar{z}}{\bar{z} - \bar{z}^*} d\phi$$

$$\frac{z^*}{z^* - z} - \frac{\bar{z}}{\bar{z} - \bar{z}^*} = \frac{z^* \bar{z}^* - z\bar{z}}{(z^* - \bar{z})(z^* - \bar{z})}$$

$$\therefore f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z^*) \frac{z^* \bar{z}^* - z\bar{z}}{(z^* - z)(\bar{z}^* - \bar{z})} d\phi$$

$$\therefore u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(R, \phi) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$

② $$\frac{z^* + z}{z^* - z} = \frac{1 + (z/z^*)}{1 - (z/z^*)} = \left(1 + \frac{z}{z^*}\right) \sum_{n=0}^{\infty} \left(\frac{z}{z^*}\right)^n = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{z}{z^*}\right)^n$$

$$z = re^{i\theta}, \quad z^* = Re^{i\phi}$$

$$\therefore \mathrm{Re}\left(\frac{z}{z^*}\right)^n = \mathrm{Re}\left[\frac{r^n}{R^n} e^{in\theta} e^{-in\phi}\right] = \left(\frac{r}{R}\right)^n \cos(n\theta - n\phi)$$

$$= \left(\frac{r}{R}\right)^n (\cos n\theta \cos n\phi + \sin n\theta \sin n\phi)$$

$$\therefore \mathrm{Re}\,\frac{z^*+z}{z^*-z}=1+2\sum_{n=1}^{\infty}\frac{r^n}{R^n}(\cos n\theta\ \cos n\phi+\sin n\theta\ \sin n\phi)$$

$$\therefore u(r,\theta)=a_o+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(a_n\cos n\theta+b_n\sin\theta)$$

四、結論

1. 複變數之基本性質不僅可應用於解析函數、三角函數、雙曲線函數、指數函數、對數函數，亦可作保角寫像，線性轉換及積分。
2. 複變數之數列和級數有冪級數，泰勒級數及勞倫級數，其剩餘值亦可積分。
3. 複變解析函數可應用於靜電場，二維空間流體運動，在電學及流體力學之應用相當廣泛。

摘 要

在工程數學的領域裡，大體上可分爲三大部分，即微分方程、線性代數和複變數等；要充分瞭解工程數學之任一大部分之性質、理論分析及其應用範圍，皆非短期內可一蹴而成的，爲此本文嘗試以複變數之性質、理論分析、應用範圍逐一介紹，俾供初學者從概略之瞭解裡，引發學習之興趣，進而推論更廣泛之應用。

本文經由理論分析結果，可獲致下列結論：

1. 複變數之基本性質不僅可應用於解析函數、三角函數、雙曲線函數、指數函數、對數函數，亦可作保角寫像，線性轉換及積分。
2. 複變數之數列和級數有冪級數、泰勒級數及勞倫級數，其剩餘值亦可積分。
3. 複變解析函數可應用於靜電場，二維空間流體運動；在電學及流體力學之應用相當廣泛。

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